

p -JETS OF p -ISOGENIES

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ABSTRACT. p -jets of finite flat maps of schemes are generally neither finite nor flat. This phenomenon can be seen already in the case of p -isogenies of group schemes. However, for p -divisible groups, this pathology tends to disappear mod p “in the limit”. We illustrate this in the case of the p -divisible groups of \mathbb{G}_m , of elliptic curves, and of formal groups of finite height; in the case of ordinary elliptic curves the structure in the limit depends on the Serre-Tate parameter.

1. INTRODUCTION

Let p be an odd prime and let $R = \widehat{\mathbb{Z}_p^{ur}}$ be the p -adic completion of the maximum unramified extension of the ring \mathbb{Z}_p of p -adic numbers. (Throughout the paper the symbol $\widehat{}$ means p -adic completion.) Let $k = R/pR$ be the residue field of R . Then for each integer $n \geq 0$ a functor J^n was introduced in [2] that attaches to any scheme of finite type X/R a (Noetherian) p -adic formal scheme $J^n(X)$ over R called the p -jet space of X of order n . For each X there are morphisms $J^n(X) \rightarrow J^{n-1}(X)$, $n \geq 1$, functorial in X , and $J^0(X) = \widehat{X}$. We refer to [2, 3] for an exposition of the theory and for some of the applications of these spaces; see also [4, 1] for a several prime version of the theory.

The functors J^n behave nicely on smooth schemes and étale morphisms: in particular if X is smooth over R then $J^n(X)$ are p -adic completions of smooth schemes over R ; and if $X \rightarrow Y$ is an étale morphism then $J^n(X) \simeq J^n(Y) \times_Y X$. So, in particular, if $X \rightarrow Y$ is finite and étale then the map $J^n(X) \rightarrow J^n(Y)$ is again finite and étale. However if $X \rightarrow Y$ is, say, finite and flat then $J^n(X) \rightarrow J^n(Y)$ is generally neither finite nor flat. This basic pathology can be seen, in its simplest form, for X a finite flat group scheme of degree p^ν over $Y = \text{Spec } R$, $\nu \geq 1$; or for p -isogenies $X \rightarrow Y$ (i.e. isogenies of degree p^ν) between smooth group schemes over R . So a typical context for these phenomena is that of p -divisible groups ($X_\nu; \nu \geq 1$). The present paper offers an analysis of p -jet spaces of such groups. The main moral of the story will be that although the p -jet spaces of the individual X_ν 's are generally highly pathological order tends to be restored “in the limit”.

Let us recall/introduce some basic notation. For any scheme of finite type X/R the rings of global functions $\mathcal{O}^n(X) := \mathcal{O}(J^n(X))$ form an inductive system; its direct limit is denoted by $\mathcal{O}^\infty(X)$. We typically denote by an upper bar reduction mod p ; in particular we set

$$(1.1) \quad \overline{\mathcal{O}^n(X)} := \mathcal{O}^n(X) \otimes_R k, \quad \overline{\mathcal{O}^\infty(X)} := \mathcal{O}^\infty(X) \otimes_R k.$$

The morphisms $\overline{\mathcal{O}^n(X)} \rightarrow \overline{\mathcal{O}^\infty(X)}$ are generally not injective (although they are injective for X/R smooth [2]). It turns out that for non-smooth X/R a special role

is then played by the rings:

$$(1.2) \quad \widetilde{\mathcal{O}^n(X)} := \text{Im}(\overline{\mathcal{O}^n(X)} \rightarrow \overline{\mathcal{O}^\infty(X)}).$$

We will be mainly interested in the case when $X = G[p^\nu]$ are the kernels of the multiplication by p^ν on $G = \mathbb{G}_m$ (the multiplicative group), $G = E$ (an elliptic curve), or $G = \mathcal{F}$ (a formal group of finite height). We illustrate in this Introduction the case of ordinary elliptic curves and that of formal groups of finite height. (The remaining cases are special cases of these cases.)

So let first E/R be an elliptic curve with ordinary reduction and Serre-Tate parameter $q = q(E) \in U_1 = 1 + pR$; cf. [6]. Also set $U_m = 1 + p^m R = U_1^{p^{m-1}}$, $m \geq 1$. Let

$$\theta \in \mathbb{Q}_p/\mathbb{Z}_p = \bigcup_{\nu=1}^{\infty} p^{-\nu}\mathbb{Z}/\mathbb{Z}$$

and assume $\nu_0 \geq 1$ is minimal with the property that $\theta \in p^{-\nu_0}\mathbb{Z}/\mathbb{Z}$. For all $\nu \geq \nu_0$ let $E[p^\nu]$ be the kernel of the multiplication by p^ν in E , viewed as an extension of $p^{-\nu}\mathbb{Z}/\mathbb{Z}$ by μ_{p^ν} , and let $E[p^\nu]_\theta$ be the connected component of $E[p^\nu]$ lying over θ . Also let $n \geq 1$. The following is part of Theorem 7.4 in the body of the paper:

Theorem 1.1.

- 1) If $q \notin U_{\nu_0+1}$ and $\theta \neq 0$ then $\mathcal{O}^n(\widetilde{E[p^\nu]_\theta}) = 0$.
- 2) If $q \in U_{\nu_0+1}$ or $\theta = 0$ then

$$\mathcal{O}^n(\widetilde{E[p^\nu]_\theta}) \simeq \frac{k[x, x', x'', \dots, x^{(n)}]}{(x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu})}.$$

Here $x, x', \dots, x^{(n)}$ are variables and the isomorphisms in the theorem are compatible, in the obvious sense, with varying n and ν . The statement of the theorem above should be contrasted with the fact that, as we shall see, for all $n, \nu \geq 1$ the rings $\overline{\mathcal{O}^n(E[p^\nu]_0)}$ have positive Krull dimension (actually they are polynomial rings in $\min\{n, \nu\}$ variables over some explicit local Artin rings).

Similarly let \mathcal{F} be a formal group law (in one variable) of finite height and let $\mathcal{F}[p^\nu]$ be the kernel of the multiplication by p^ν viewed as a finite flat group scheme over R . Let $n \geq 1$. The following is part of Theorem 8.3 in the body of the paper:

Theorem 1.2.

$$\mathcal{O}^n(\widetilde{\mathcal{F}[p^\nu]}) \simeq \frac{k[x, x', x'', \dots, x^{(n)}]}{(x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu})}.$$

Again we have compatibility with n, ν varying; also for $n, \nu \geq 1$, $\overline{\mathcal{O}^n(\mathcal{F}[p^\nu])}$ have positive Krull dimension (and, again, they are polynomial rings in $\min\{n, \nu\}$ variables over some explicit local Artin rings).

Remark 1.3. It would be interesting to have a generalization of Theorems 1.1 and 1.2 to the case of arbitrary p -divisible groups.

Remark 1.4. It is interesting to note the following phenomenon. Let X_0, \dots, X_ν be closed subschemes of the affine line \mathbb{A}^1 over R which are, say, finite and flat over R , and let

$$X = \bigcup_{i=0}^{\nu} X_i$$

(scheme theoretic union inside \mathbb{A}^1 , defined by the intersection of the defining ideals). Then, in general,

$$J^n(X) \neq \bigcup_{i=0}^{\nu} J^n(X_i)$$

as closed subschemes of $J^n(\mathbb{A}^1)$. An example is provided by the case when $X_i = \text{Spec } R[\zeta_{p^i}]$ where ζ_{p^i} is a p^i -th root of unity. In this case $J^n(X_0) = \text{Spec } R$ and, as we shall see later in the paper, $J^n(X_i) = \emptyset$ for $i \geq 1$ and $n \geq 1$; on the other hand $X = \mu_{p^\nu}$ (kernel of multiplication by p^ν on \mathbb{G}_m over R) and hence $J^n(\mu_{p^\nu})$ has a non-reduced reduction mod p by Theorem 1.2. It would be interesting to understand this phenomenon more generally when, for instance, $X_i = \text{Spec } R[\alpha_i]$ with α_i integers in a finite ramified extension of the fraction field of R .

A few words about proofs and the structure of the paper. The structure of the rings in Theorems 1.1 and 1.2 is deceptively simple. What actually happens is that the equations defining the p -jet spaces (of arbitrary order) of finite p -group schemes (even of the simplest ones such as μ_{p^ν}) are a priori extremely complex. The trick to handle this complexity (and to eventually reveal the simple formulae in the above theorems) is to introduce certain filtrations on the relevant rings and ideals such that the Fermat quotient operator controlling the structure of p -jet spaces is, in its turn, controlled by these filtrations in a way that allows induction arguments to work. So after recalling from [2, 3] some of the basic concepts we shall be dealing with we will introduce the above mentioned filtrations and then, in subsequent sections, we will investigate the p -jets of the divisible groups of \mathbb{G}_m , ordinary elliptic curves, formal groups of finite height, and supersingular elliptic curves respectively. The \mathbb{G}_m case will be used as a step in the analysis of all the other cases. Also, the section on formal groups will contain more general results about p -jet spaces of schemes defined by iterates of certain rather general power series that do not necessarily come from formal groups.

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2. REVIEW OF SOME BASIC CONCEPTS [2, 3]

Rings in this paper will always be assumed commutative with unity. A p -derivation $\delta : A \rightarrow A$ on a ring A is a set theoretic map satisfying

$$\begin{aligned} \delta(x+x) &= \delta x + \delta y + C_p(x, y) \\ \delta(xy) &= x^p \delta y + y^p \delta x + p \delta x \delta y, \end{aligned}$$

where C_p is the polynomial:

$$C_p(X, Y) = p^{-1}(X^p + Y^p - (X + Y)^p) \in \mathbb{Z}[X, Y].$$

If δ is as above then $\phi : A \rightarrow A$, $\phi(x) = x^p + p \delta x$, is a ring homomorphism. Note that $\delta(xy) = x^p \delta y + \phi(y) \delta x = y^p \delta x + \phi(x) \delta y$. Also δ and ϕ commute. If A is p -torsion free then δ is, of course, uniquely determined by ϕ ; also

$$(2.1) \quad \delta(x_1 + \dots + x_m) = \delta x_1 + \dots + \delta x_m + C_p(x_1, \dots, x_m),$$

where

$$C_p(X_1, \dots, X_m) := p^{-1} \left(\sum_{i=1}^m X_i^p - \left(\sum_{i=1}^m X_i \right)^p \right) \in \mathbb{Z}[X_1, \dots, X_m].$$

Now the ring $R = \widehat{\mathbb{Z}_p^{ur}}$ has a unique p -derivation defined by $\delta x = (\phi(x) - x^p)/p$ where $\phi : R \rightarrow R$ is the unique ring automorphism lifting the p -power Frobenius on R/pR . Let x be a variable (or more generally an N -tuple of variables x_1, \dots, x_N .) We consider the δ -polynomial ring $R\{x\} = R[x, x', x'', \dots]$; this is the polynomial ring in variables $x, x', x'', \dots, x^{(n)}, \dots$, where x', x'', \dots are variables (or N -tuples of variables), equipped with the unique p -derivation $\delta : R\{x\} \rightarrow R\{x\}$ such that $\delta x = x'$, $\delta x' = x''$, etc. For X a scheme of finite type over R one defines the p -jet spaces $J^n(X)$, $n \geq 0$ [2]. The latter are p -adic formal schemes over R fitting into a projective system

$$\dots \rightarrow J^n(X) \rightarrow J^{n-1}(X) \rightarrow \dots \rightarrow J^0(X) = \widehat{X}.$$

Note that $X \mapsto J^n(X)$ are functors commuting with open immersions and more generally with étale maps in the sense that if $X \rightarrow Y$ is étale then $J^n(X) \simeq J^n(Y) \times_Y X$ in the category of p -adic formal schemes. If $X = \text{Spec } R[x]/(f)$ for a tuple of variables x and a tuple of polynomials f and then

$$J^n(X) = \text{Spf } R[x, x', \dots, x^{(n)}] \frown (f, \delta f, \dots, \delta^n f).$$

In particular if $Y \rightarrow X$ is a closed immersion so is $J^n(Y) \rightarrow J^n(X)$ for all n . Moreover J^n commutes with fiber products: $J^n(Y \times_X Z) \simeq J^n(Y) \times_{J^n(X)} J^n(Z)$. The rings $\mathcal{O}^n(X) := \mathcal{O}(J^n(X))$ form an inductive system and their direct limit $\mathcal{O}^\infty(X)$ has a natural p -derivation δ induced by the p -derivation δ on $R\{x\}$. The following universality property holds. Assume for simplicity X is affine. Then any R -algebra homomorphism of $\mathcal{O}(X)$ into a p -adically complete ring B equipped with a p -derivation δ is induced by a unique R -algebra homomorphism $\mathcal{O}^\infty(X) \rightarrow B$ that commutes with δ .

3. FILTRATIONS

In this section we will introduce and study some basic filtrations, especially on rings equipped with p -derivations. In the next section we will specialize to the case when the ring in question is the ring $R\{x\}$ of δ -polynomials.

Let A be a ring which for simplicity we assume p -torsion free and let I be an ideal in A . For any integer $\nu \geq 0$ we denote by $I^{[\nu]}$ the ideal generated by all the elements of the form $p^i f^{p^j}$ with $f \in I$, $i, j \geq 0$, $i + j = \nu$. In particular $I^{[1]} = I$.

Lemma 3.1.

- 1) $I^{[p^{\nu+1}]} \subset I^{[\nu]}$.
- 2) If $f \in I^{[\nu]}$ then $pf \in I^{[p^{\nu+1}]}$.
- 3) If $f \in I^{[\nu]}$ then $f^p \in I^{[p^{\nu+1}]}$.
- 4) If I is generated by a family $\{f_s; s \in S\}$ then $I^{[p^{\nu}]}$ is generated by the family $\{p^i f_s^{p^j}; s \in S, i, j \geq 0, i + j = \nu\}$.

Proof. Assertions 1 and 2 are clear. For assertion 3 if $f = \sum_{s=1}^N p^{i_s} f_s^{p^{j_s}} g_s$ with $f_s \in I$, $g_s \in A$, and $i_s + j_s = \nu$ for all s then

$$(3.1) \quad f^p \in \sum_{s=1}^N (p^{i_s} f_s^{p^{j_s}} g_s)^p + p C_p(p^{i_1} f_1^{p^{j_1}} g_1, \dots, p^{i_N} f_N^{p^{j_N}} g_N) \subset I^{[p^{\nu+1}]}.$$

To check assertion 4 it is sufficient to prove that if $g \in I$ then g^{p^t} is in the ideal generated by the family $\{p^i f_s^{p^j}; s \in S, i, j \geq 0, i + j = t\}$. One proves this by induction on $t \geq 0$. The case $t = 0$ is clear. Now if the statement is true for $t = \nu$ and we set $f = g^{p^\nu}$ then we are done by equation (3.1). \square

In what follows we assume we are given a p -derivation $\delta : A \rightarrow A$.

Lemma 3.2. *Assume $\delta(I) \subset I$. Then:*

- 1) $\delta(I^{[p^\nu]}) \subset I^{[p^\nu]}$;
- 2) $\phi(I^{[p^\nu]}) \subset I^{[p^{\nu+1}]}$.

Proof. First, for any $f \in A$, we have the following computation:

$$\begin{aligned}
 \delta(f^{p^\nu}) &= \frac{1}{p}(\phi(f^{p^\nu}) - f^{p^{\nu+1}}) \\
 (3.2) \quad &= \frac{1}{p}((f^p + p\delta f)^{p^\nu} - f^{p^{\nu+1}}) \\
 &= p^\nu f^{p(p^\nu-1)} \delta f + p^{\nu+1} (\delta f)^2 P(f, \delta f)
 \end{aligned}$$

where P is a polynomial with \mathbb{Z} -coefficients; indeed this is because for $2 \leq m \leq p^\nu$ we have

$$p^{\nu+1} \mid \binom{p^\nu}{m} p^{m-1}.$$

In particular if $f \in I$ then $\delta(f^{p^\nu}) \in p^\nu I \subset I^{[p^\nu]}$.

Let's prove assertion 1. In view of the equation (2.1) it is enough to note that for $i + j = \nu$, $f \in I$, $g \in A$ we have $\delta(p^i f^{p^j} g) \in I^{[p^\nu]}$. Now

$$\begin{aligned}
 \delta(p^i f^{p^j} g) &= \delta(p^i) f^{p^{j+1}} g^p + p^i \delta(f^{p^j} g) \\
 &= \delta(p^i) f^{p^{j+1}} g^p + p^i (\delta(f^{p^j}) g^p + f^{p^{j+1}} \delta g + p(\delta(f^{p^j}))(\delta g)) \\
 &\in I^{[p^\nu]},
 \end{aligned}$$

by Lemma 3.1 and because $\delta(p^i)$ is either 0 or in $p^{i-1}A$ according as $i = 0$ or $i \geq 1$.

To prove assertion 2 note that if $f \in I^{[p^\nu]}$ then $\phi(f) = f^p + p\delta f \in I^{[p^{\nu+1}]}$ by Lemma 3.1. \square

In what follows we assume we are given, in addition, a filtration on A

$$A^0 \subset A^1 \subset A^2 \subset \dots \subset A^n \subset \dots \subset A,$$

by subrings A^n such that $\delta A^n \subset A^{n+1}$ for all $n \geq 0$. Then we define a new filtration by subrings

$$(3.3) \quad A^{\{0\}} \subset A^{\{1\}} \subset A^{\{2\}} \subset \dots \subset A^{\{n\}} \subset \dots \subset A,$$

$$A^{\{n\}} := \sum_{s=0}^{\infty} p^s A^{n+s} = A^n + pA^{n+1} + p^2 A^{n+2} + \dots$$

Lemma 3.3.

- 1) $pA^{\{n+1\}} \subset A^{\{n\}}$;
- 2) $\delta(A^{\{n\}}) \subset A^{\{n+1\}}$;
- 3) $\phi(A^{\{n\}}) \subset A^{\{n\}}$.

Proof. A trivial exercise. \square

We will also need the following general:

Lemma 3.4. *For any $f, g \in A^0$ we have the following equality of ideals in A^n :*

$$(f - g, \delta(f - g), \dots, \delta^n(f - g)) = (f - g, \delta f - \delta g, \dots, \delta^n f - \delta^n g).$$

Proof. Induction on n . The induction step follows from the congruence

$$\begin{aligned} \delta(\delta^{n-1}f - \delta^{n-1}g) - (\delta^n f - \delta^n g) &= C_p(\delta^{n-1}f, -\delta^{n-1}g) \\ &\equiv C_p(\delta^{n-1}g, -\delta^{n-1}g) \pmod{\delta^{n-1}f - \delta^{n-1}g} \\ &= 0. \end{aligned}$$

□

4. p -JETS OF p -ISOGENIES OF \mathbb{G}_m

In this section we specialize the discussion of the previous section to the case when

$$(4.1) \quad A = R\{x\}, \quad A^n = R[x, x', \dots, x^{(n)}], \quad I = (x', x'', \dots).$$

So in this case, explicitly,

$$A^{\{n\}} = R[x, x', \dots, x^{(n)}] + pR[x, x', \dots, x^{(n+1)}] + p^2R[x, x', \dots, x^{(n+2)}] + \dots,$$

while $I^{[p^\nu]}$ is the ideal of $R\{x\}$ generated by all δ -polynomials of the form $p^i(x^{(s)})^{p^j}$, with $s \geq 1, i, j \geq 0, i + j = \nu$; cf. assertion 4 in Lemma 3.1.

Here is the main result of this section:

Lemma 4.1. *For all integers $n, \nu \geq 1$ we have*

$$\delta^n(x^{p^\nu}) \in \begin{cases} p^{\nu-n+1}x^{p^n(p^\nu-1)}\phi^{n-1}(x') + (p^{\nu-n+2}A^{\{1\}}) \cap I^{[p^\nu]}, & \text{if } n \leq \nu + 1 \\ x^{p^n(p^\nu-1)}\phi^\nu(x^{(n-\nu)}) + A^{\{n-\nu-1\}} \cap I^{[p^\nu]}, & \text{if } n \geq \nu + 2. \end{cases}$$

Proof. First note that for $n \leq \nu + 1$ we have $\phi^{n-1}(x') \in I^{[p^{n-1}]}$ by Lemma 3.2 and hence

$$(4.2) \quad p^{\nu-n+1}x^{p^n(p^\nu-1)}\phi^{n-1}(x') \in I^{[p^\nu]},$$

by Lemma 3.1. Similarly, for $n \geq \nu + 2$, we have

$$(4.3) \quad x^{p^n(p^\nu-1)}\phi^\nu(x^{(n-\nu)}) \in I^{[p^\nu]},$$

by Lemma 3.2. We also claim that

$$(4.4) \quad \delta^n(x^{p^\nu}) \in I^{[p^\nu]}.$$

To check (4.4) it is enough, by Lemma 3.2, to check that $\delta(x^{p^\nu}) \in I^{[p^\nu]}$; this however follows from equation (3.2). In view of (4.2), (4.3), (4.4), in order to prove our theorem it is enough to prove that

$$\delta^n(x^{p^\nu}) \in \begin{cases} p^{\nu-n+1}x^{p^n(p^\nu-1)}\phi^{n-1}(x') + p^{\nu-n+2}A^{\{1\}}, & \text{if } n \leq \nu + 1 \\ x^{p^n(p^\nu-1)}\phi^\nu(x^{(n-\nu)}) + A^{\{n-\nu-1\}}, & \text{if } n \geq \nu + 2. \end{cases}$$

We fix ν and proceed by induction on $n \geq 1$. For $n = 1$ we are done by (3.2). Next assume the Lemma is true for n and we prove it for $n + 1$.

Assume first $n \leq \nu + 1$. By Lemma 3.2 $\phi^{n-1}(x') \in A^{\{1\}}$. So we have

$$\begin{aligned} \delta^{n+1}(x^{p^\nu}) &\in \delta(p^{\nu-n+1}x^{p^n(p^\nu-1)}\phi^{n-1}(x')) + \delta(p^{\nu-n+2}A^{\{1\}}) \\ &\quad + C_p(p^{\nu-n+1}A^{\{1\}}, p^{\nu-n+2}A^{\{1\}}). \end{aligned}$$

Clearly the last term in the last equation is in $p^{\nu-n+1}A^{\{1\}}$. Also, by Lemma 3.3:

$$\begin{aligned}\delta(p^{\nu-n+2}A^{\{1\}}) &\subset \delta(p^{\nu-n+2})A^{\{1\}} + p^{\nu-n+2}\delta(A^{\{1\}}) \\ &\subset p^{\nu-n+1}A^{\{1\}} + p^{\nu-n+2}A^{\{2\}} \\ &\subset p^{\nu-n+1}A^{\{1\}}.\end{aligned}$$

Now for $n \leq \nu$ we have

$$\delta(p^{n-\nu+1}) = p^{n-\nu} - p^{p(n-\nu+1)-1} \in p^{n-\nu} - p^{n-\nu+1}\mathbb{Z}$$

hence:

$$\begin{aligned}\delta(p^{\nu-n+1}x^{p^n(p^\nu-1)}\phi^{n-1}(x')) &= \delta(p^{\nu-n+1})x^{p^{n+1}(p^\nu-1)}(\phi^{n-1}(x'))^p \\ &\quad + p^{\nu-n+1}\delta(x^{p^n(p^\nu-1)}\phi^{n-1}(x')) \\ &\in p^{\nu-n}x^{p^{n+1}(p^\nu-1)}\phi^{n-1}((x')^p) + p^{\nu-n+1}A^{\{1\}} \\ &\quad + p^{\nu-n+1}\delta(x^{p^n(p^\nu-1)}(\phi^{n-1}(x'))^p) \\ &\quad + p^{\nu-n+1}\phi(x^{p^n(p^\nu-1)})\delta(\phi^{n-1}(x')) \\ &\subset p^{\nu-n}x^{p^{n+1}(p^\nu-1)}\phi^{n-1}((x')^p) \\ &\quad + p^{\nu-n+1}A^{\{1\}} \\ &\quad + p^{\nu-n}(x^{p^{n+1}(p^\nu-1)} + pA^1)(\phi^{n-1}(px'')) \\ &\subset p^{\nu-n}x^{p^{n+1}(p^\nu-1)}\phi^n(x') + p^{\nu-n+1}A^{\{1\}}\end{aligned}$$

because $\delta \circ \phi^{n-1} = \phi^{n-1} \circ \delta$, $(x')^p + px'' = \phi(x')$, and

$$p^{\nu-n} \cdot pA^1 \cdot (\phi^{n-1}(px'')) \subset p^{\nu-n+1} \cdot A^1 \cdot pA^{\{2\}} \subset p^{\nu-n+1}A^{\{1\}}.$$

So for $n \leq \nu$ we get

$$\delta^{n+1}(x^{p^\nu}) = p^{\nu-n}x^{p^{n+1}(p^\nu-1)}\phi^n(x') + p^{\nu-n+1}A^{\{1\}},$$

which ends the induction step in case $n \leq \nu$.

For $n = \nu + 1$ we get

$$\begin{aligned}\delta(p^{\nu-n+1}x^{p^n(p^\nu-1)}\phi^{n-1}(x')) &= \delta(x^{p^n(p^\nu-1)}\phi^{n-1}(x')) \\ &= \delta(x^{p^n(p^\nu-1)})(\phi^{n-1}(x'))^p \\ &\quad + \phi(x^{p^n(p^\nu-1)})\delta(\phi^{n-1}(x')) \\ &\in A^{\{1\}} + (x^{p^{n+1}(p^\nu-1)} + pA^1)(\phi^{n-1}(x'')) \\ &= x^{p^{n+1}(p^\nu-1)}\phi^{n-1}(x'') + A^{\{1\}},\end{aligned}$$

by Lemma 3.3. Hence

$$\delta^{n+1}(x^{p^\nu}) = x^{p^{n+1}(p^\nu-1)}\phi^\nu(x^{(n+1-\nu)}) + A^{\{n-\nu\}},$$

which ends the induction step in case $n = \nu + 1$.

Assume now $n \geq \nu + 2$; then, by Lemma 3.3,

$$\begin{aligned}\delta^{n+1}(x^{p^\nu}) &= \delta(x^{p^n(p^\nu-1)}\phi^\nu(x^{(n-\nu)})) + \delta(A^{\{n-\nu-1\}}) \\ &\quad + C_p(A^{\{n-\nu\}}, A^{\{n-\nu-1\}}) \\ &\in \delta(x^{p^n(p^\nu-1)}\phi^\nu(x^{(n-\nu)})) + A^{\{n-\nu\}} \\ &= x^{p^{n+1}(p^\nu-1)}\delta(\phi^\nu(x^{(n-\nu)})) \\ &\quad + \phi^{\nu+1}(x^{(n-\nu)})\delta(x^{p^n(p^\nu-1)}) + A^{\{n-\nu\}} \\ &= x^{p^{n+1}(p^\nu-1)}\phi^\nu(x^{(n+1-\nu)}) \\ &\quad + A^{\{n-\nu\}} \cdot pA^1 + A^{\{n-\nu\}} = A^{\{n-\nu\}}.\end{aligned}$$

This ends the induction step in case $n \geq \nu + 2$. \square

Corollary 4.2.

$$\delta^n(x^{p^\nu}) \in \begin{cases} A^{\{0\}} \cap I^{[p^\nu]} & \text{if } n \leq \nu, \\ A^{\{n-\nu\}} \cap I^{[p^\nu]} & \text{if } n \geq \nu + 1. \end{cases}$$

For any element $f \in A^n$ we denote by $\bar{f} \in \bar{A}^n := \overline{A^n} = k[x, x', \dots, x^{(n)}]$ the class of $f \bmod p$. In particular one can consider the reduction mod p of the ideal I , $\bar{I} = (x', x'', \dots) \subset \bar{A} = k[x, x', x'', \dots]$. Then

$$\overline{I^{[p^\nu]}} = \bar{I}^{[p^\nu]} = ((x')^{p^\nu}, (x'')^{p^\nu}, \dots).$$

Moreover clearly

$$\bar{A}^n \cap \bar{I}^{[p^\nu]} = ((x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu}) \subset \bar{A}^n.$$

Also for any $f \in A^{\{n\}} \cap I^{[p^\nu]}$ we have $\bar{f} \in \bar{A}^n \cap \bar{I}^{[p^\nu]}$.

Corollary 4.3. *The element $\overline{\delta^n(x^{p^\nu})} \in \bar{A}^n$ satisfies*

$$\overline{\delta^n(x^{p^\nu})} = \begin{cases} 0, & \text{if } 1 \leq n \leq \nu, \\ x^{p^{\nu+1}(p^\nu-1)}(x')^{p^\nu}, & \text{if } n = \nu + 1, \\ x^{p^n(p^\nu-1)}(x^{(n-\nu)})^{p^\nu} + (\text{element in } \bar{A}^{n-\nu-1} \cap \bar{I}^{[p^\nu]}), & \text{if } n \geq \nu + 2. \end{cases}$$

Example 4.4. *The “smallest” interesting case is $\nu = 1$, $n = 3$,*

$$\overline{\delta^3(x^p)} = x^{p^3(p-1)}(x'')^p - \frac{1}{2}x^{p^3(p-2)}(x')^{2p}.$$

The following will also be useful later.

Lemma 4.5.

- 1) $\delta^n(x^p) \in R[x^p, x', \dots, x^{(n)}]$ for $n \geq 0$.
- 2) $\delta^n(x_1 x_2) \in R[x_1^p, x_2^p, x_1', x_2', \dots, x_1^{(n)}, x_2^{(n)}]$.

Proof. Trivial induction on n . □

The natural context for Lemma 4.1 is as follows. Recall the multiplicative group scheme

$$\mathbb{G}_m = \text{Spec } R[x, x^{-1}]$$

over R and the p -isogeny

$$[p^\nu]_{\mathbb{G}_m} : \mathbb{G}_m \rightarrow \mathbb{G}_m$$

given at the level of rings by $x \mapsto x^{p^\nu}$. The p -jet spaces of \mathbb{G}_m are given by

$$J^n(\mathbb{G}_m) = \text{Spf } R[x, x^{-1}, x', \dots, x^{(n)}]^\wedge.$$

Consider the multiplication by p^ν map

$$[p^\nu]_{J^n(\mathbb{G}_m)} : J^n(\mathbb{G}_m) \rightarrow J^n(\mathbb{G}_m).$$

By the universality property of J^n the map $[p^\nu]_{J^n(\mathbb{G}_m)}$ coincides with the map $J^n([p^\nu]_{\mathbb{G}_m})$ induced by $[p^\nu]_{\mathbb{G}_m}$. So the map $[p^\nu]_{J^n(\mathbb{G}_m)} = J^n([p^\nu]_{\mathbb{G}_m})$ is given at the level of rings by

$$(4.5) \quad [p^\nu]^* : \mathcal{O}^n(\mathbb{G}_m) = R[x, x^{-1}, x', \dots, x^{(n)}]^\wedge \rightarrow R[x, x^{-1}, x', \dots, x^{(n)}]^\wedge,$$

$$x \mapsto x^{p^\nu}, \quad x' \mapsto \delta(x^{p^\nu}), \quad \dots, \quad x^{(n)} \mapsto \delta^n(x^{p^\nu}).$$

Lemma 4.1 can be seen then as a description of the map $[p^\nu]_{J^n(\mathbb{G}_m)} = J^n([p^\nu]_{\mathbb{G}_m})$. Note that the map (4.5) is neither finite nor flat; indeed its reduction mod p

$$(4.6) \quad \overline{[p^\nu]^*} : \overline{\mathcal{O}^n(\mathbb{G}_m)} = k[x, x^{-1}, x', \dots, x^{(n)}] \rightarrow k[x, x^{-1}, x', \dots, x^{(n)}]$$

is neither finite nor flat because it is non-injective. See the Corollary 4.6 below and also Proposition 5.8 for more on this. Nevertheless note that by Corollary 4.3 we have induced homomorphisms

$$k[x, x^{-1}, x^{(\nu+1)}, \dots, x^{(n)}] \simeq \frac{k[x, x^{-1}, x', \dots, x^{(n)}]}{(x', \dots, x^{(\nu)})} \rightarrow k[x, x^{-1}, x', \dots, x^{(n-\nu)}],$$

for $n \geq \nu + 1$, and these homomorphisms are finite algebra maps. By a dimension argument these homomorphisms must be injective. Since finite maps of non-singular varieties are automatically flat ([5], p. 461) we have:

Corollary 4.6. *For $n \geq \nu + 1$ there are induced injective finite flat maps*

$$\overline{[p^\nu]^*} : \overline{\mathcal{O}^n(\mathbb{G}_m)} / (x', \dots, x^{(\nu)}) \rightarrow \overline{\mathcal{O}^{n-\nu}(\mathbb{G}_m)}.$$

5. p -JETS OF μ_{p^ν}

Start again with the multiplicative group \mathbb{G}_m and the isogeny $[p^\nu]_{\mathbb{G}_m} : \mathbb{G}_m \rightarrow \mathbb{G}_m$. Let

$$\mu_{p^\nu} = \mathbb{G}_m[p^\nu] := \text{Ker}([p^\nu]_{\mathbb{G}_m}) = \text{Spec } R[x, x^{-1}] / (x^{p^\nu} - 1) = \text{Spec } R[x] / (x^{p^\nu} - 1)$$

be the kernel of $[p^\nu]_{\mathbb{G}_m}$. More generally, for any $a \in R^\times$, one may consider the finite flat scheme

$$\mu_{p^\nu}^a := \text{Spec } R[x] / (x^{p^\nu} - a).$$

Its functor of points is given by $\mu_{p^\nu}^a(S) = \{s \in S; s^{p^\nu} = a\}$ for any R -algebra S . Then $\mu_{p^\nu}^a$ has a natural structure of μ_{p^ν} -torsor. More generally, for any $a, b \in R^\times$ we have a natural morphism

$$\mu_{p^\nu}^a \times \mu_{p^\nu}^b \xrightarrow{\text{can}} \mu_{p^\nu}^{ab}$$

given on S -points by $(s, t) \mapsto st$. We also have a natural isomorphism

$$\mu_{p^\nu}^{a^{p^\nu} b} \xrightarrow{\text{can}} \mu_{p^\nu}^b$$

given on points by $s \mapsto s/a$. Recall that we have set $U_m = 1 + p^m R = U_1^{p^{m-1}}$. So if $a \in U_{\nu+1}$ then $a = b^{p^\nu}$, $b \in U_1$, so division by b gives an isomorphism

$$\mu_{p^\nu}^a \simeq \mu_{p^\nu}.$$

Finally the system $(\mu_{p^\nu}; \nu \geq 1)$ is a p -divisible group with embeddings $\mu_{p^\nu} \subset \mu_{p^{\nu+1}}$ given by the inclusions on points. More generally for any $a \in R^\times$ and any $\nu_0 \geq 1$ the schemes $(\mu_{p^{\nu-\nu_0}}^a; \nu \geq \nu_0)$, form an inductive system with embeddings

$$(5.1) \quad \mu_{p^{\nu_0}}^a \subset \mu_{p^{\nu_0+1}}^{a^p} \subset \dots \subset \mu_{p^\nu}^{a^{p^{\nu-\nu_0}}} \subset \mu_{p^{\nu+1}}^{a^{p^{\nu+1-\nu_0}}} \subset \dots$$

given by the inclusions on points. Note that if $a \in U_{\nu_0+1}$ then we can write $a = b^{p^{\nu_0}}$ and hence division by b gives an isomorphism between the inductive system (5.1) and the inductive system

$$(5.2) \quad \mu_{p^{\nu_0}} \subset \mu_{p^{\nu_0+1}} \subset \dots \subset \mu_{p^\nu} \subset \mu_{p^{\nu+1}} \subset \dots$$

Recall from Remark ?? that $J^n(\mathbb{G}_m) = \text{Spf } R[x, x^{-1}, x', \dots, x^{(n)}]^\wedge$ and that $[p^\nu]_{J^n(\mathbb{G}_m)} : J^n(\mathbb{G}_m) \rightarrow J^n(\mathbb{G}_m)$ is given at the level of rings by $x \mapsto x^{p^\nu}$, $x' \mapsto \delta(x^{p^\nu})$, etc. By the commutation of J^n with fiber products it follows that

$$J^n(\mu_{p^\nu}) = \text{Ker}(J^n([p^\nu]_{\mathbb{G}_m})) = \text{Ker}([p^\nu]_{J^n(\mathbb{G}_m)}) =: J^n(\mathbb{G}_m)[p^\nu].$$

More generally, if $a \in R^\times$, and if we still denote by $a : \text{Spec } R \rightarrow \mathbb{G}_m$ the point defined by $x \mapsto a$ then

$$J^n(\mu_{p^\nu}^a) = J^n([p^\nu]_{\mathbb{G}_m}^{-1}(a)) = (J^n([p^\nu]_{\mathbb{G}_m}))^{-1}(J^n(a)) = ([p^\nu]_{J^n(\mathbb{G}_m)})^{-1}(J^n(a))$$

where $J^n(a) : \text{Spec } R \rightarrow J^n(\mathbb{G}_m)$ is given, at the level of rings, by

$$x \mapsto a, \quad x' \mapsto \delta a, \quad \dots, \quad x^{(n)} \mapsto \delta^n a.$$

It follows that

Proposition 5.1. *We have:*

$$\mathcal{O}^n(\mu_{p^\nu}^a) = \frac{R[x, x', \dots, x^{(n)}]^\wedge}{(x^{p^\nu} - a, \delta(x^{p^\nu}) - \delta a, \dots, \delta^n(x^{p^\nu}) - \delta^n a)}.$$

In particular

$$\mathcal{O}^n(\mu_{p^\nu}) = \frac{R[x, x', \dots, x^{(n)}]^\wedge}{(x^{p^\nu} - 1, \delta(x^{p^\nu}), \dots, \delta^n(x^{p^\nu}))}.$$

Alternatively Proposition 5.1 follows from Lemma 3.4.

Let $J^n(\mu_{p^\nu})_1$ be the kernel of the projection $J^n(\mu_{p^\nu}) \rightarrow J^0(\mu_{p^\nu}) = \widehat{\mu_{p^\nu}}$ and write $\mathcal{O}^n(\mu_{p^\nu})_1 := \mathcal{O}(J^n(\mu_{p^\nu})_1)$. Then we have the following splitting result:

Proposition 5.2. *We have an isomorphism*

$$\mathcal{O}^n(\mu_p) \simeq \mathcal{O}(\mu_p) \widehat{\otimes} \mathcal{O}^n(\mu_p)_1$$

compatible with the group laws. Equivalently we have

$$J^n(\mu_p) = \widehat{\mu_p} \times J^n(\mu_p)_1$$

as groups in the category of formal p -adic schemes over R .

Proof. Let $\delta^i(x^p) = F_i(x^p, x', \dots, x^{(i)})$; cf. Lemma 4.5. Then

$$\begin{aligned} \mathcal{O}^n(\mu_p)_1 &= R[x, x', \dots, x^{(n)}]^\wedge / (x - 1, F_1(x^p, x'), \dots, F_n(x^p, x', \dots, x^{(n)})) \\ &= R[x', \dots, x^{(n)}]^\wedge / (F_1(1, x'), \dots, F_n(1, x', \dots, x^{(n)})) \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{O}^n(\mu_p) &= R[x, x', \dots, x^{(n)}]^\wedge / (x^p - 1, F_1(x^p, x'), \dots, F_n(x^p, x', \dots, x^{(n)})) \\ &= R[x, x', \dots, x^{(n)}]^\wedge / (x^p - 1, F_1(1, x'), \dots, F_n(1, x', \dots, x^{(n)})) \\ &\simeq (R[x] / (x^p - 1)) \widehat{\otimes} (R[x', \dots, x^{(n)}]^\wedge / (F_1(1, x'), \dots, F_n(1, x', \dots, x^{(n)}))) \\ &= \mathcal{O}(\mu_p) \widehat{\otimes} \mathcal{O}^n(\mu_p)_1. \end{aligned}$$

The fact that the above isomorphism is compatible with group laws follows from the fact that, in view of Lemma 4.5, we have

$$\delta^n(x_1 x_2) \equiv \delta^n(x_1 x_2)|_{x_1=x_2=1} \mod (x_1^p - 1, x_2^p - 1).$$

□

Remark 5.3. It would be interesting to understand what happens with Proposition 5.2 if we consider μ_{p^ν} instead of μ_p .

Proposition 5.4. *For all $n \geq 1$ we have*

$$\lim_{\leftarrow \nu} \mathcal{O}^n(\mu_{p^\nu})_1 = R[x', \dots, x^{(n)}]^\wedge.$$

Proof. By Proposition 5.1

$$\mathcal{O}^n(\mu_{p^\nu})_1 = \frac{R[x', \dots, x^{(n)}]^\wedge}{(\delta(x^{p^\nu})|_{x=1}, \dots, \delta(x^{p^\nu})|_{x=1})}.$$

Since, by Lemma 4.1 the denominator in the last equation is in $(p^{\nu-n+1})$ we are done by the following general well known Lemma. \square

Lemma 5.5. *If A is a Noetherian ring, I is an ideal, A is I -adically complete, and (L_n) is a descending sequence of ideals such that $L_n \subset I^n$ then $A = \lim_{\leftarrow} A/L_n$.*

Proof. We recall the argument. The map $A \rightarrow \lim_{\leftarrow} A/L_n$ is clearly injective. It is surjective because if $f_n \in A$, $f_{n+1} - f_n \in L_n$ then $f_{n+1} - f_n \in I^n$ hence there exists $f \in A$ such that $f - f_n \in I^n$. Now fix m ; since for $n \geq m$, $f - f_m = (f - f_n) + (f_n - f_m) \in I^n + L_m$ and (by [9], Theorems 8.2 and 8.14) $\cap_{n \geq 1} (I^n + L_m) = L_m$ we get $f - f_m \in L_m$. \square

According to our general notation (Equations 1.1 and 1.2) we next recall the rings

$$\overline{\mathcal{O}^n(\mu_{p^\nu}^a)}, \quad \widetilde{\mathcal{O}^n(\mu_{p^\nu}^a)}.$$

Also recall we set $U_m = 1 + p^m R = U_1^{p^{m-1}}$, $m \geq 1$.

Theorem 5.6. *Let $n, \nu \geq 1$ and $a \in U_1$.*

- 1) *If $a \notin U_{\nu+1}$ then $\mathcal{O}^n(\mu_{p^\nu}^a) = 0$.*
- 2) *If $a \in U_{\nu+1}$ then*

$$(5.3) \quad \widetilde{\mathcal{O}^n(\mu_{p^\nu}^a)} \simeq \widetilde{\mathcal{O}^n(\mu_{p^\nu})} \simeq \frac{k[x, x', x'', \dots, x^{(n)}]}{((x-1)^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu})}.$$

$$(5.4) \quad \overline{\mathcal{O}^n(\mu_{p^\nu}^a)} \simeq \overline{\mathcal{O}^n(\mu_{p^\nu})} \simeq \frac{k[x, x', x'', \dots, x^{(n)}]}{(x^{p^\nu} - 1)} \quad \text{if } n \leq \nu,$$

$$(5.5) \quad \overline{\mathcal{O}^n(\mu_{p^\nu}^a)} \simeq \overline{\mathcal{O}^n(\mu_{p^\nu})} \simeq \frac{k[x, x', x'', \dots, x^{(n)}]}{(x^{p^\nu} - 1, (x')^{p^\nu}, \dots, (x^{(n-\nu)})^{p^\nu})} \quad \text{if } n \geq \nu + 1.$$

Proof. To prove assertion 1) let $a \in U_m \setminus U_{m+1}$, $1 \leq m \leq \nu$. Then $\delta^m a \in R^\times$ hence, by Corollary 4.3, the reduction mod p of $\delta^m(x^{p^\nu}) - \delta^m a$ is in k^\times . By Proposition 5.1 $\overline{\mathcal{O}^N(\mu_{p^\nu}^a)} = 0$ for all $N \geq m$ and hence $\widetilde{\mathcal{O}^n(\mu_{p^\nu}^a)} = 0$ for all $n \geq 1$.

To prove assertion 2) note that the equalities (7.7) and (7.8) follow from Corollary 4.3 and Proposition 5.1; in particular we have

$$\lim_{\leftarrow m} \overline{\mathcal{O}^m(\mu_{p^\nu})} = \frac{k[x, x', x'', \dots]}{(x^{p^\nu} - 1, (x')^{p^\nu}, (x'')^{p^\nu}, \dots)}.$$

Now (7.6) follows from the fact that the intersection

$$k[x, x', \dots, x^{(n)}] \cap (x^{p^\nu} - 1, (x')^{p^\nu}, (x'')^{p^\nu}, \dots)$$

in the ring $k[x, x', x'', \dots]$ equals the ideal

$$(x^{p^\nu} - 1, (x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu}).$$

□

Corollary 5.7. *Let $n, \nu_0 \geq 1$, $a \in U_1$. Then*

$$(5.6) \quad \lim_{\leftarrow \nu} \mathcal{O}^n(\widetilde{\mu_{p^\nu}^{a_{p^\nu-\nu_0}}}) = \begin{cases} 0 & \text{if } a \notin U_{\nu_0+1} \\ k[[x-1, x', \dots, x^{(n-1)}]] & \text{if } a \in U_{\nu_0+1} \end{cases}$$

We also remark the following:

Proposition 5.8. *Assume $n, \nu \geq 1$. Then*

- 1) $\mathcal{O}^n(\mu_{p^\nu})$ is not a finite R -algebra.
- 2) $\mathcal{O}^n(\mu_{p^\nu})$ is not a flat R -algebra.

Proof. If $\mathcal{O}^n(\mu_{p^\nu})$ is a finite R -algebra then $\overline{\mathcal{O}^n(\mu_{p^\nu})}$ is a finite k -algebra which is not the case, cf. equations (7.7) and (7.8). If $\mathcal{O}^n(\mu_{p^\nu})$ is a flat R -algebra then it is torsion free. Since, by equation (3.2), $\delta(x^{p^\nu}) \in p^\nu x^{p(p^\nu-1)}x' + p^{\nu+1}A^1$ it follows that an element in $x' + pA^1$ is zero in $\mathcal{O}^n(\mu_{p^\nu})$ hence x' is zero in $\overline{\mathcal{O}^n(\mu_{p^\nu})}$ hence in $\widetilde{\mathcal{O}^n(\mu_{p^\nu})}$. Hence x' is in the denominator of the ring in equation (7.6), a contradiction. □

Remark 5.9. The author is indebted to A. Saha for pointing out 2) in Proposition 5.8 above.

6. p -JETS OF THE IRREDUCIBLE COMPONENTS OF μ_{p^ν}

Next note that μ_{p^ν} is connected and has $\nu + 1$ irreducible components:

$$\mu_{p^\nu} = \bigcup_{i=0}^{\nu} \mu_{p^\nu, i}, \quad \mu_{p^\nu, i} := \text{Spec } R[\zeta_{p^i}],$$

where ζ_{p^i} is a primitive p -root of unity. So $\zeta_1 = 1$, $R[\zeta_1] = R = R[x]/(x-1)$, and

$$R[\zeta_{p^i}] = R[x]/(\Phi_{p^i}(x)), \quad \Phi_{p^i}(x) := \frac{x^{p^i} - 1}{x^{p^{i-1}} - 1}, \quad i \geq 2.$$

The scheme theoretic intersection of these components is

$$\text{Spec } R[x]/(x-1, p) = \text{Spec } k.$$

In deep contrast with Proposition 5.6 the p -jets of these components are completely uninteresting:

Proposition 6.1.

- 1) $\mathcal{O}^n(\mu_{p^\nu, 0}) = R$ for $n \geq 1$;
- 2) $\mathcal{O}^n(\mu_{p^\nu, i}) = 0$ for $i \geq 1$ and $n \geq 1$.

Proof. The first equality is clear. The second follows from the fact that $\Phi_{p^i}(x+1)$ is an Eisenstein polynomial plus the following general Proposition 6.2 below. □

Proposition 6.2. *Let $f(x) = x^e + a_1x^{e-1} + \dots + a_{e-1}x + a_e \in R[x]$ be an Eisenstein polynomial (i.e. $e \geq 2$, $a_1, \dots, a_e \in pR$, $a_e \notin p^2R$) and let $X = \text{Spec } R[x]/(f(x))$. Then $\mathcal{O}^n(X) = 0$ for $n \geq 1$.*

Proof. It is enough to show that $\delta f(x)$ is invertible in $\mathcal{O}^1(X) \otimes k$. Now we have:

$$\begin{aligned} \delta f(x) &= \delta(x^e) + \delta(a_1 x^{e-1}) + \dots + \delta(a_{e-1} x) + \delta a_e + C_p(x^e, a_1 x^{e-1}, \dots, a_e) \\ &\equiv e x^{p(e-1)} x' + x^{p(e-1)} (\delta a_1) + \dots + x^p (\delta a_{e-1}) + \delta a_e \pmod{p}. \end{aligned}$$

Since the image of x in $\mathcal{O}^1(X) \otimes k$ is nilpotent and the image of δa_e in the same ring is invertible it follows that the image of $\delta f(x)$ in this ring is invertible which ends the proof. \square

7. p -JETS OF $E[p^\nu]$ FOR ORDINARY ELLIPTIC CURVES

We start with a review of extensions of $p^{-\nu}\mathbb{Z}/\mathbb{Z}$ by μ_{p^ν} . For any group (respectively group scheme) G we denote by $G[N]$ the kernel of the multiplication by N map. For a finite group Γ we continue to denote by Γ the étale group scheme over R attached to Γ ; so for any connected R -algebra S , $\Gamma(S) = \Gamma$. In particular we have the connected R -group scheme $\mu_{p^\nu} = \mathbb{G}_m[p^\nu]$. Also one can consider the étale R -group scheme $p^{-\nu}\mathbb{Z}/\mathbb{Z}$. Let $R_m = R/p^m R$, $m \geq 1$. We also view, when appropriate, μ_{p^ν} and $p^{-\nu}\mathbb{Z}/\mathbb{Z}$ as R_m -group schemes via base change. Then, by “Kummer theory”,

$$\begin{aligned} \text{Ext}_{R_m}^1(p^{-\nu}\mathbb{Z}/\mathbb{Z}, \mu_{p^\nu}) &\simeq R_m^\times / (R_m^\times)^{p^\nu} \\ (7.1) \quad &\simeq (1 + pR_m) / (1 + pR_m)^{p^\nu} \\ &\simeq (1 + pR_m) / (1 + p^{\nu+1}R_m). \end{aligned}$$

We will need the following explicit description of the above isomorphism. Let $q \in 1 + pR$. Consider the finite flat R -scheme

$$\Gamma_{p^\nu}^q = \prod_{i=0}^{p^\nu-1} \mu_{p^\nu}^{q^i}.$$

This is a group scheme with multiplication given by

$$\mu_{p^\nu}^{q^i} \times \mu_{p^\nu}^{q^j} \xrightarrow{\text{can}} \mu_{p^\nu}^{q^{i+j}} \xrightarrow{\text{can}} \mu_{p^\nu}^{q^l},$$

where $0 \leq l < p^\nu$, $i + j \equiv l \pmod{p^\nu}$. The functor of points of $\Gamma_{p^\nu}^q$ is given by

$$\Gamma_{p^\nu}^q(S) = \{(s, i); s \in S^\times, 0 \leq i < p^\nu, s^{p^\nu} = q^i\}$$

for any R -algebra S ; the multiplication on points is given by $(s, i) \cdot (t, j) = (st, i + j)$ if $i + j < p^\nu$ and $(s, i) \cdot (t, j) = (st/q, i + j - p^\nu)$ if $i + j \geq p^\nu$. We have an extension

$$(7.2) \quad 0 \rightarrow \mu_{p^\nu} \rightarrow \Gamma_{p^\nu}^q \rightarrow p^{-\nu}\mathbb{Z}/\mathbb{Z} \rightarrow 0$$

(the second map being given on points by $(s, i) \mapsto \frac{i}{p^\nu} + \mathbb{Z}$). The following is “well known”:

Lemma 7.1. *The isomorphism (7.1) is given by attaching to the class of $q \in 1 + pR$ in $(1 + pR_m) / (1 + p^{\nu+1}R_m)$ the class of the extension (7.2) in $\text{Ext}_{R_m}^1(p^{-\nu}\mathbb{Z}/\mathbb{Z}, \mu_{p^\nu})$.*

Note that the system $(\Gamma_{p^\nu}^q; \nu \geq 1)$ is a p -divisible group via the morphisms $\Gamma_{p^\nu}^q \rightarrow \Gamma_{p^{\nu+1}}^q$ given on points by $(s, i) \mapsto (s, pi)$ and given on schemes by the inclusions $\mu_{p^\nu}^{q^i} \subset \mu_{p^{\nu+1}}^{q^{pi}}$. The p -divisible group $(\Gamma_{p^\nu}^q; \nu \geq 1)$ is an extension of the p -divisible group $(p^{-\nu}\mathbb{Z}/\mathbb{Z}; \nu \geq 1)$ by the p -divisible group $(\mu_{p^\nu}; \nu \geq 1)$, where the latter are viewed as p -divisible groups with respect to the natural inclusions.

Next we consider an elliptic curve E/R . References for this are [10, 6]. Let \overline{E}/k be its reduction mod p and let E^{for} be the formal group attached to E . (We use the superscript *for* rather than \wedge because the latter is used in the present paper to denote p -adic completion). Let $E^{for}[p^\nu]$ be the kernel of the multiplication by p^ν on E^{for} , viewed as a finite flat group scheme over R . Assume in what follows that \overline{E} is ordinary. Then

$$(7.3) \quad E^{for} \simeq \mathbb{G}_m^{for};$$

we fix such an isomorphism. So we have induced isomorphisms $E^{for}[p^\nu] \simeq \mu_{p^\nu}$. Moreover we fix isomorphisms

$$(7.4) \quad \overline{E}(k)[p^\nu] \simeq \mathbb{Z}/p^\nu\mathbb{Z} \simeq p^{-\nu}\mathbb{Z}/\mathbb{Z}.$$

With the isomorphisms (7.3) and (7.4) fixed one defines the Serre-Tate parameter $q = q(E) \in 1 + pR$ of E as follows. The isomorphisms (7.4) define a basis (α_ν) of the Tate module $T_p\overline{E} = \varprojlim \overline{E}(k)[p^\nu]$, $\alpha_\nu \in \overline{E}(k)[p^\nu]$ a generator, $p\alpha_\nu = \alpha_{\nu-1}$. If $A_\nu \in E(R)$ lifts α_ν then one defines the Serre-Tate parameter $q(E) \in 1 + pR$ as the image of $\lim p^\nu A_\nu \in E^{for}(R)$ via the isomorphism $E^{for}(R) \simeq 1 + pR$ induced by (7.3); cf. [6], section 2. On the other hand with the isomorphisms (7.3) and (7.4) fixed there are induced exact sequences of finite flat group schemes over R :

$$(7.5) \quad 0 \rightarrow \mu_{p^\nu} \rightarrow E[p^\nu] \rightarrow p^{-\nu}\mathbb{Z}/\mathbb{Z} \rightarrow 0.$$

Cf., say, [6]. Also by loc.cit. we have

Lemma 7.2. *The class of the extension (7.5) is the image of the Serre-Tate parameter $q(E) \in 1 + pR$ under the isomorphism (7.1).*

We conclude by Lemmas 7.1 and 7.2 that if $q = q(E)$ then $E[p^\nu]$ and $\Gamma_{p^\nu}^q$ are isomorphic as extensions over R_m for any m ; the isomorphisms are compatible as m varies so we get the following:

Corollary 7.3. *$E[p^\nu]$ and $\Gamma_{p^\nu}^q$ are isomorphic as extensions over R for $q = q(E)$. In particular if $0 \leq i < p^\nu$ and $\theta = \frac{i}{p^\nu} + \mathbb{Z} \in p^{-\nu}\mathbb{Z}/\mathbb{Z}$ then the connected component $E[p^\nu]_\theta$ of $E[p^\nu]$ lying above θ is isomorphic to $\mu_{p^\nu}^{q^i}$. Consequently if we fix ν_0 and an index $0 \leq i_0 < p^{\nu_0}$ and if $\theta = \frac{i_0}{p^{\nu_0}} + \mathbb{Z} \in p^{-\nu_0}\mathbb{Z}/\mathbb{Z}$ then the inductive system*

$$E[p^{\nu_0}]_\theta \subset E[p^{\nu_0+1}]_\theta \subset E[p^{\nu_0+2}]_\theta \subset \dots \subset E[p^\nu]_\theta \subset \dots$$

identifies with the inductive system

$$\mu_{p^{\nu_0}}^{q^{i_0}} \subset \mu_{p^{\nu_0+1}}^{(q^{i_0})^p} \subset \mu_{p^{\nu_0+2}}^{(q^{i_0})^{p^2}} \subset \dots \subset \mu_{p^\nu}^{(q^{i_0})^{p^{\nu-\nu_0}}} \subset \dots$$

Putting together Theorem 5.6 and Corollary 7.3 (and making the change of variables $x \mapsto x + 1$) we get:

Theorem 7.4. *Let E/R be an elliptic curve with ordinary reduction and Serre-Tate parameter $q = q(E) \in U_1$. Let $n, \nu_0 \geq 1$, $\theta \in p^{-\nu_0}\mathbb{Z}/\mathbb{Z}$, and assume ν_0 is minimal with this property. Let $E[p^\nu]_\theta$ be the connected component of $E[p^\nu]$ lying over θ . Then:*

- 1) *If $q \notin U_{\nu_0+1}$ and $\theta \neq 0$ then $\mathcal{O}^n(\widetilde{E[p^\nu]_\theta}) = 0$.*
- 2) *If $q \in U_{\nu_0+1}$ or $\theta = 0$ then*

$$(7.6) \quad \mathcal{O}^n(\widetilde{E[p^\nu]_\theta}) \simeq \frac{k[x, x', x'', \dots, x^{(n)}]}{(x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu})}.$$

$$(7.7) \quad \overline{\mathcal{O}^n(E[p^\nu]_\theta)} \simeq \frac{k[x, x', x'', \dots, x^{(n)}]}{(x^{p^\nu})} \quad \text{if } n \leq \nu,$$

$$(7.8) \quad \overline{\mathcal{O}^n(E[p^\nu]_\theta)} \simeq \frac{k[x, x', x'', \dots, x^{(n)}]}{(x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n-\nu)})^{p^\nu})} \quad \text{if } n \geq \nu + 1.$$

Corollary 7.5. *Let $n, \nu_0 \geq 1$, $a \in U_1$. Then*

$$(7.9) \quad \lim_{\leftarrow \nu} \mathcal{O}^n(\widetilde{E[p^\nu]_\theta}) = \begin{cases} 0 & \text{if } q \notin U_{\nu_0+1} \text{ and } \theta \neq 0 \\ k[[x, x', \dots, x^{(n-1)}]] & \text{if } q \in U_{\nu_0+1} \text{ or } \theta = 0. \end{cases}$$

Also Corollaries 5.8 and 7.3 imply

Corollary 7.6. *Let $n, \nu \geq 1$. Then the map*

$$J^n([p^\nu]_E) = [p^\nu]_{J^n(E)} : J^n(E) \rightarrow J^n(E)$$

is neither finite nor flat.

An analogue of Corollary 4.6 should hold nevertheless in the elliptic case as well.

8. p -JETS OF $\mathcal{F}[p^\nu]$

Recall that a series $F(x) \in xR[[x]]$ without constant term is said to have finite height if $F \not\equiv 0 \pmod{p}$; if this is the case the height of F is defined as the largest integer $h \geq 0$ such that $F \in R[[x^{p^h}]] + pR[[x]]$.

Remark 8.1. The main example we have in mind here arises as follows. Consider a formal group law $\mathcal{F} \in R[[x_1, x_2]]$. By [10] the multiplication by p in \mathcal{F} is given by a series $F(x) := [p]_{\mathcal{F}}(x)$ satisfying $F(x) \equiv px \pmod{x^2}$. The height of \mathcal{F} is defined to be the height of $F(x)$; if the height is finite then it is ≥ 1 . Not every series of height ≥ 1 which is $\equiv px \pmod{x^2}$ is the multiplication by p of a formal group law \mathcal{F} ; indeed if $F(x) = [p]_{\mathcal{F}}$ has height h then one knows that the x -adic valuation of the reduction mod p of F in $k[[x]]$ is exactly p^h ; cf. [10], p.127.

Let $F(x) \in xR[[x]]$ be a series of finite height $h \geq 1$ and let $F^{\circ \nu} = F \circ \dots \circ F$ be the ν fold composition of F with itself for $\nu \geq 1$. Let ep^h be the x -adic valuation of the reduction mod p of F . (So if $F(x) = [p]_{\mathcal{F}}(x)$ for some formal group law \mathcal{F} then $e = 1$.) By “Weierstrass preparation” (cf. [7], p. 130) $F^{\circ \nu} = U_\nu \cdot P_\nu$ where $U_\nu \in R[[x]]^\times$ and $P_\nu \in R[x]$ is monic of degree $e^\nu p^{h\nu}$, $P_\nu \equiv x^{e^\nu p^{h\nu}} \pmod{p}$. Consider the scheme:

$$X_\nu := \operatorname{Spec} \frac{R[[x]]}{(F^{\circ \nu})} = \operatorname{Spec} \frac{R[[x]]}{(P_\nu)} \simeq \operatorname{Spec} \frac{R[x]}{(P_\nu)};$$

the latter isomorphism follows from “Euclid division” by P_ν in $R[[x]]$; cf. [7], p. 129. So X_ν is a finite flat scheme over R of degree $e^\nu p^{h\nu}$ and we have a natural sequence of closed immersions

$$(8.1) \quad X_1 \subset X_2 \subset \dots \subset X_\nu \subset \dots$$

Note that if $F(x) = [p]_{\mathcal{F}}(x)$ is the multiplication by p on some formal group law \mathcal{F} then $X_\nu = \mathcal{F}[p^\nu]$, where the latter is the kernel of $[p^\nu]_{\mathcal{F}}$ on \mathcal{F} and indeed the inductive system (8.1) coincides with the p -divisible group

$$\mathcal{F}[p] \subset \mathcal{F}[p^2] \subset \dots \subset \mathcal{F}[p^\nu] \subset \dots$$

of \mathcal{F} ; cf. [11]. According to our general notation in equations 1.1 and 1.2 we consider the rings

$$(8.2) \quad \overline{\mathcal{O}^n(X_\nu)}, \quad \widetilde{\mathcal{O}^n(X_\nu)}.$$

Remark 8.2. Recall (cf. [8], p. 480) that any formal group law \mathcal{F} over R of height $h = 1$ is isomorphic to the multiplicative formal group law hence in particular $\mathcal{F}[p^\nu] \simeq \mu_{p^\nu}$. Hence Theorem 5.6 provides a computation of the rings (8.2) in this case. The following Theorem covers the case of formal group laws \mathcal{F} of height $h \geq 2$. More generally the Theorem below treats the case of iterates of series of height ≥ 1 which are not necessarily coming from formal groups; so even for height 1 the Theorem below is not covered by Theorem 5.6.

Theorem 8.3. *Let $F(x) \in xR[[x]]$ be a series of finite height $h \geq 1$ satisfying $F(x) \equiv px \pmod{x^2}$. For all $\nu \geq 1$ consider the scheme $X_\nu := \text{Spec } \frac{R[[x]]}{(F^{\circ \nu})}$. Then we have:*

$$(8.3) \quad \widetilde{\mathcal{O}^n(X_\nu)} = \frac{k[x, x', \dots, x^{(n)}]}{(x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu})} \quad \text{if } n \geq 1,$$

$$(8.4) \quad \overline{\mathcal{O}^n(X_\nu)} = \frac{k[x, x', \dots, x^{(n)}]}{(x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n-\nu)})^{p^\nu})} \quad \text{if } n \geq \nu,$$

$$(8.5) \quad \overline{\mathcal{O}^n(X_\nu)} = \frac{k[x, x', \dots, x^{(n)}]}{(x^{p^\mu})} \quad \text{if } 1 \leq n \leq \nu - 1,$$

where $\mu \geq \nu$.

Remark 8.4. Let \mathcal{F} be a formal group law over R of finite height h . Note that the morphism

$$\overline{\mathcal{O}(\mathcal{F}[p^\nu])} \rightarrow \overline{\mathcal{O}^n(\mathcal{F}[p^\nu])}$$

is injective if $h = 1$ (cf. Theorem 5.6) but not injective if the height $h \geq 2$ and $n \geq \nu$ (cf. Theorem 8.3).

Remark 8.5. To prove the theorem above note first that:

$$(8.6) \quad \begin{aligned} \mathcal{O}^n(X_\nu) &= R[x, x', \dots, x^{(n)}]^\wedge / (P_\nu, \delta P_\nu, \dots, \delta^n P_\nu) \\ &= R[[x]][x', \dots, x^{(n)}]^\wedge / (P_\nu, \delta P_\nu, \dots, \delta^n P_\nu) \\ &= R[[x]][x', \dots, x^{(n)}]^\wedge / (F^{\circ \nu}, \delta(F^{\circ \nu}), \dots, \delta^n(F^{\circ \nu})). \end{aligned}$$

So we will be concerned from now on with understanding the structure of the expressions $\delta^i(F^{\circ \nu})$. To do this we need to develop some filtration machinery on power series.

We start by considering the decreasing filtration of $\mathcal{A}^0 := R[[x]]$ by the subrings \mathcal{A}_ν^0 , $\nu \geq 1$, defined by

$$\mathcal{A}_\nu^0 = R[[x^{p^\nu}]] + pR[[x^{p^{\nu-1}}]] + p^2R[[x^{p^{\nu-2}}]] + \dots + p^\nu R[[x]] \subset R[[x]].$$

Let v_p be the p -adic valuation on R .

Lemma 8.6.

- 1) $\mathcal{A}_\nu^0 = \{\sum_{n \geq 0} a_n x^n \in R[[x]] ; v_p(a_n) \geq \nu - v_p(n)\}$.
- 2) If $G_1, G_2, G_3, \dots \in \mathcal{A}_\nu^0$, $G_m \in x^m R[[x]]$. Then $\sum_{m \geq 1} G_m \in \mathcal{A}_\nu^0$.
- 3) If $H \in \mathcal{A}_\nu^0$, $H(0) = 0$, and $G \in R[[x]]$ then $G(H(x)) \in \mathcal{A}_\nu^0$.
- 4) $p\mathcal{A}_\nu^0 \subset \mathcal{A}_{\nu+1}^0$.
- 5) If $G \in \mathcal{A}_\nu^0$ then $G^p \in \mathcal{A}_{\nu+1}^0$.
- 6) If $F \in \mathcal{A}_1^0$ and $F(0) = 0$ then $F^{\circ \nu} \in \mathcal{A}_\nu^0$.

Proof. 1) is easy. 2) clearly follows from 1). 3) clearly follows from 2). 4) and 5) are clear. 6) follows from 3), 4), 5). \square

We continue by considering the filtration

$$\mathcal{A}^n = R[[x]][x', \dots, x^{(n)}]^\wedge, \quad n \geq 0$$

on

$$\mathcal{A} := \bigcup_{n \geq 0} \mathcal{A}^n.$$

(Here $\mathcal{A}^0 = R[[x]]$.) There is a natural p -derivation δ on \mathcal{A} sending $\delta x = x'$, $\delta x' = x''$, etc. Note that $\delta \mathcal{A}^n \subset \mathcal{A}^{n+1}$ for all n . So according to equation (3.3) we may then consider the filtration $\mathcal{A}^{\{n\}}$ on \mathcal{A} . Finally let

$$\mathcal{J} = (x, x', x'', \dots) \subset \mathcal{A}.$$

So we may consider the descending filtration of \mathcal{J} by ideals $\mathcal{J}^{[p^\nu]}$, $\nu \geq 0$. Note that with A^n, A, I as in 4.1 we have $A^n \subset \mathcal{A}^n$, $A \subset \mathcal{A}$, $I \subset \mathcal{J}$, and hence $I^{[p^\nu]} \subset \mathcal{J}^{[p^\nu]}$.

Lemma 8.7. Let $n, i \geq 1$.

$$\delta^n(p^i x) \in \begin{cases} p^{i-n} \phi^n(x) + (p^{i-n+1} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{i+1}]} & \text{if } n \leq i \\ \phi^i(x^{(n-i)}) + \mathcal{A}^{\{n-i-1\}} \cap \mathcal{J}^{[p^{i+1}]} & \text{if } n \geq i+1 \end{cases}$$

Proof. Induction on n . The case $n = 1$ is clear. Now assume the above is true for some $n \geq 1$. If $n \leq i-1$ then by Lemma 3.2 $p^{i-n} \phi(x)^p \in p^{i-n} \mathcal{J}^{[p^{n+1}]} \subset \mathcal{J}^{[p^{i+1}]}$ so

$$\begin{aligned} \delta^{n+1}(p^i x) &\in \delta(p^{i-n} \phi^n(x)) + \delta((p^{i-n+1} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{i+1}]}) \\ &\quad + C_p(p^{i-n} \mathcal{A}^{\{0\}}, (p^{i-n+1} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{i+1}]}) \\ &\subset p^{i-n-1} \phi^{n+1}(x) - p^{(i-n)p-1} \phi^n(x)^p + (p^{i-n} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{i+1}]} \\ &\quad + (p^{i-n+1} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{i+1}]} \\ &\subset p^{i-n-1} \phi^{n+1}(x) + (p^{i-n} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{i+1}]}. \end{aligned}$$

If $n \geq i+1$ we have

$$\begin{aligned} \delta^{n+1}(p^i x) &\in \delta(\phi^i(x^{(n-i)})) + \delta(\mathcal{A}^{\{n-i-1\}} \cap \mathcal{J}^{[p^{i+1}]}) \\ &\quad + C_p(\mathcal{A}^{\{n-i\}}, \mathcal{A}^{\{n-i-1\}} \cap \mathcal{J}^{[p^{i+1}]}) \\ &\in \phi^i(x^{(n+1-i)}) + \mathcal{A}^{\{n-i\}} \cap \mathcal{J}^{[p^{i+1}]}. \end{aligned}$$

The case $n = i$ is similar. \square

Lemma 8.8. Let $G \in xR[[x]]$ be any series without constant term. Then for all $n \geq 1$:

$$\delta^n(G(x)) \in \left(\frac{dG}{dx}\right)^{p^n} x^{(n)} + \mathcal{A}^{\{n-1\}} \cap \mathcal{J}^{[p]}.$$

Proof. We proceed by induction on n . To check the statement for $n = 1$ write $G(x) = \sum_{n \geq 1} a_n x^n$; then

$$\begin{aligned} \delta(G(x)) &= \frac{1}{p} [\sum_{n \geq 1} \phi(a_n)(x^p + px')^n - (\sum_{n \geq 1} a_n x^n)^p] \\ &\subset x^p \mathcal{A}^0 + (\sum_{n \geq 1} n a_n x^{n-1})^p x' + px' \mathcal{A}^1 \\ &\subset \left(\frac{dG}{dx}\right)^p x' + \mathcal{A}^{\{0\}} \cap \mathcal{J}^{[p]}, \end{aligned}$$

which settles the case $n = 1$. For the induction step, assuming the statement true for some $n \geq 1$, we have

$$\begin{aligned} \delta^{n+1}(G(x)) &\in \delta\left(\left(\frac{dG}{dx}\right)^{p^n} x^{(n)}\right) + \delta(\mathcal{A}^{\{n-1\}} \cap \mathcal{J}^{[p]}) + C_p(\mathcal{A}^{\{n\}}, \mathcal{A}^{\{n-1\}} \cap \mathcal{J}^{[p]}) \\ &\in \left(\frac{dG}{dx}\right)^{p^{n+1}} x^{(n+1)} + \phi(x^{(n)}) \cdot \delta\left(\left(\frac{dG}{dx}\right)^{p^n}\right) + \mathcal{A}^{\{n\}} \cap \mathcal{J}^{[p]}. \end{aligned}$$

Now, using Theorem 4.1, we have

$$\begin{aligned} \phi(x^{(n)}) \cdot \delta\left(\left(\frac{dG}{dx}\right)^{p^n}\right) &\in (\mathcal{A}^n \cap \mathcal{J}) \cdot p^n \mathcal{A}^1 \\ &\subset \mathcal{A}^n \cap p^n \mathcal{J} \\ &\subset \mathcal{A}^{\{n\}} \cap \mathcal{J}^{[p]} \end{aligned}$$

and we are done. \square

Next let $\Sigma = \Sigma(x) \in xR[[x]]$ be any series without constant term and consider the unique ring endomorphism $\Sigma^* : \mathcal{A} \rightarrow \mathcal{A}$ such that $\Sigma^* x = \Sigma(x)$, $\Sigma^* x' = \delta(\Sigma(x))$, $\Sigma^* x'' = \delta^2(\Sigma(x))$, etc. Clearly $\Sigma^*(\mathcal{A}^n) \subset \mathcal{A}^n$ for $n \geq 0$ and hence $\Sigma^*(\mathcal{A}^{\{n\}}) \subset \mathcal{A}^{\{n\}}$ for $n \geq 0$. It is trivial to see that Σ^* and δ commute on \mathcal{A} ; similarly Σ^* and ϕ commute on \mathcal{A} . Moreover for any two series Σ_1, Σ_2 we have the following compatibility of upper $*$ with composition: $(\Sigma_1 \circ \Sigma_2)^* = \Sigma_2^* \circ \Sigma_1^*$.

For any integer $a \in \mathbb{Z}$ write $a^+ = \max\{a, 0\}$.

Lemma 8.9. *Assume $\Sigma(x) = x^{p^m}$, $m \geq 1$, $\nu \geq 0$, $n \geq 0$; then:*

- 1) $\Sigma^* \mathcal{J}^{[p^\nu]} \subset \mathcal{J}^{[p^{\nu+m}]}$;
- 2) $\Sigma^* \mathcal{A}^{\{n\}} \subset \mathcal{A}^{\{(n-m)^+\}}$.

Proof. To check 1) it is enough to check it for $m = 1$. Now 1) follows from the following computations in which $i + j = \nu$:

$$\Sigma^*(p^i(x^{(s)})^{p^j}) = p^i(\delta^s(x^p))^{p^j} \in p^i \mathcal{J}^{[p^{j+1}]} \subset \mathcal{J}^{[p^{i+j+1}]} = \mathcal{J}^{[p^{\nu+1}]};$$

in the above we used the fact that since $x^p \in \mathcal{J}^{[p]}$ we have $\delta^s(x^p) \in \mathcal{J}^{[p]}$ hence $(\delta^s(x^p))^{p^j} \in \mathcal{J}^{[p^{j+1}]}$; cf. Lemma 3.1. To prove 2) it is enough, by the compatibility with composition, to prove these two statements for $m = 1$ and $n \geq 1$ which we now assume. Now by Theorem 4.1 we have $\delta^n(x^p) \in \mathcal{A}^{\{n-1\}}$ for $n \geq 1$. Consequently, for $n \geq 1$ and $F \in \mathcal{A}^{n+i}$ we have

$$\Sigma^*(p^i F(x, \dots, x^{(n+i)})) = p^i F(x^p, \dots, \delta^{n+i}(x^p)) \in p^i \mathcal{A}^{\{n+i-1\}} \subset \mathcal{A}^{\{n-1\}}$$

which proves 2). \square

Lemma 8.10. *Let $G(x) \in xR[[x]]$, $m \geq 1$, $\nu \geq 1$. Then for $i + j = \nu$, $i, j \geq 0$ we have that $\delta^m(p^i G(x^{p^j}))$ belongs to*

$$(8.7) \quad \begin{cases} p^{i-m} \phi^m(G(x^{p^j})) + (p^{i-m+1} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{\nu+1}]}, & \text{if } m \leq i \\ \phi^i \left\{ \left(\frac{dG}{dx}(x^{p^j}) \right)^{p^{m-i}} \delta^{m-i}(x^{p^j}) \right\} + \mathcal{A}^{\{(m-\nu-1)^+\}} \cap \mathcal{J}^{[p^{\nu+1}]}, & \text{if } m \geq i+1. \end{cases}$$

Proof. Set $\Sigma(x) = x^{p^j}$. Using Lemmas 8.7, 8.8, 8.9 we have the following computation for $1 \leq m \leq i$:

$$\begin{aligned} \delta^m(p^i G(x^{p^j})) &= \delta^m(\Sigma^* G^*(p^i x)) \\ &= \Sigma^* G^*(\delta^m(p^i x)) \\ &= \Sigma^* G^*\{p^{i-m} \phi^m(x) + (p^{i-m+1} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{i+1}]} \} \\ &= \Sigma^* G^*(p^{i-m} \phi^m(x)) + (p^{i-m+1} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{i+1+j}]} \\ &= p^{i-m} \phi^m(G(x^{p^j})) + (p^{i-m+1} \mathcal{A}^{\{0\}}) \cap \mathcal{J}^{[p^{\nu+1}]}. \end{aligned}$$

For $m \geq i+1$ we have:

$$\begin{aligned} \delta^m(p^i G(x^{p^j})) &= \Sigma^* G^*(\delta^m(p^i x)) \\ &\in \Sigma^* G^*\{\phi^i(x^{(m-i)}) + \mathcal{A}^{\{m-i-1\}} \cap \mathcal{J}^{[p^{i+1}]} \} \\ &= \Sigma^* G^*\{\phi^i(x^{(m-i)})\} + \mathcal{A}^{\{(m-i-1-j)^+\}} \cap \mathcal{J}^{[p^{i+1+j}]} \\ &= \phi^i \Sigma^*(\delta^{m-i} G) + \mathcal{A}^{\{(m-\nu-1)^+\}} \cap \mathcal{J}^{[p^{\nu+1}]} \\ &= \phi^i \Sigma^*\left\{ \left(\frac{dG}{dx} \right)^{p^{m-i}} x^{(m-i)} + \mathcal{A}^{\{m-i-1\}} \cap \mathcal{J}^{[p]} \right\} \\ &\quad + \mathcal{A}^{\{(m-\nu-1)^+\}} \cap \mathcal{J}^{[p^{\nu+1}]} \\ &= \phi^i \left\{ \left(\frac{dG}{dx}(x^{p^j}) \right)^{p^{m-i}} \cdot \delta^{m-i}(x^{p^j}) \right\} + \mathcal{A}^{\{(m-\nu-1)^+\}} \cap \mathcal{J}^{[p^{\nu+1}]}. \end{aligned}$$

□

For any element $f \in \mathcal{A}^n$ we denote by $\bar{f} \in \bar{\mathcal{A}}^n := \overline{\mathcal{A}^n} = k[[x]][x', \dots, x^{(n)}]$ the class of $f \bmod p$. In particular one can consider the reduction mod p of the ideal $\mathcal{J}, \bar{\mathcal{J}} = (x, x', x'', \dots) \subset \bar{\mathcal{A}} = k[[x]][x', x'', \dots]$. Then

$$\overline{\mathcal{J}^{[p^\nu]}} = \bar{\mathcal{J}}^{[p^\nu]} = (x^{p^\nu}, (x')^{p^\nu}, (x'')^{p^\nu}, \dots).$$

Moreover clearly

$$\bar{\mathcal{A}}^n \cap \bar{\mathcal{J}}^{[p^\nu]} = (x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu}) \subset \bar{\mathcal{A}}^n.$$

Also for any $f \in \mathcal{A}^{\{n\}} \cap \mathcal{J}^{[p^\nu]}$ we have $\bar{f} \in \bar{\mathcal{A}}^n \cap \bar{\mathcal{J}}^{[p^\nu]}$. Now Lemma 8.10 trivially implies :

Lemma 8.11. *Let $m \geq 0$, $\nu \geq 1$, $i + j = \nu$, $i, j \geq 0$, $G \in xR[[x]]$. Then the element*

$$\overline{\delta^m(p^i G(x^{p^j}))} \in \bar{\mathcal{A}}^m$$

is given by

$$\begin{cases} 0, & \text{if } m \leq i-1 \\ G(x^{p^j})^{p^i}, & \text{if } m = i \\ (\text{elt. in } \bar{\mathcal{A}}^0 \cap \bar{\mathcal{J}}^{[p^{\nu+1}]})^{p^m}, & \text{if } i+1 \leq m \leq \nu \\ \left(x^{p^j-1} \frac{dG}{dx}(x^{p^j}) \right)^{p^m} (x^{(m-\nu)})^{p^\nu} + (\text{elt. in } \bar{\mathcal{A}}^{m-\nu-1} \cap \bar{\mathcal{J}}^{[p^{\nu+1}]})^{p^m}, & \text{if } m \geq \nu+1. \end{cases}$$

In particular

$$\overline{\delta^m(p^i G(x^{p^j}))} \in \overline{\mathcal{A}}^{(m-\nu)^+} \cap \overline{\mathcal{J}}^{[p^\nu]}.$$

Lemma 8.12. *Let y be an N -tuple y_1, \dots, y_N of variables. Then, for $n \geq 1$,*

$$\delta^n \left(\sum_{i=1}^N y_i \right) = \sum_{i=1}^N y_i^{(n)} + P_{N,n}(y, y', \dots, y^{(n-1)})$$

in $R\{y\}$ where $P_{N,n}$ is a polynomial with \mathbb{Z} -coefficients without constant term or linear terms.

Proof. Induction on n . □

For the next Lemma note that for any series $G(x) \in \mathcal{A}_\nu^0$ we have $\frac{dG}{dx} \in p^\nu R[[x]]$ so we may consider the series

$$\overline{p^{-\nu} \frac{dG}{dx}} \in k[[x]].$$

Then Lemmas 8.11 and 8.12 immediately imply:

Lemma 8.13. *Consider a series $G(x) = \sum_{j=0}^\nu p^{\nu-j} G_j(x^{p^j})$, $G_j(x) \in xR[[x]]$. Then, for $\nu \geq 1$ and $n \geq 0$ the element $\overline{\delta^n G} \in \overline{\mathcal{A}}^n$ is given by*

$$\overline{\delta^n G} = \begin{cases} (G_{\nu-n}(x^{p^{\nu-n}}))^{p^n} + (\text{elt. in } x^{2p^\nu} k[[x]]) & \text{if } 0 \leq n \leq \nu \\ \left(\overline{p^{-\nu} \frac{dG}{dx}} \right)^{p^n} (x^{(n-\nu)})^{p^\nu} + (\text{elt. in } \overline{\mathcal{A}}^{n-\nu-1} \cap \overline{\mathcal{J}}^{[\nu]}) & \text{if } n \geq \nu + 1. \end{cases}$$

Proof of Theorem 8.3. By assertion 6) in Lemma 8.6 we may write $F^{\circ\nu} = \sum_{j=0}^\nu p^{\nu-j} G_j(x^{p^j})$, $G_j \in R[[x]]$, $j \geq 0$. We may choose the G_j s in $xR[[x]]$ and then $G_0(x) \equiv x \pmod{x^2}$. Also $\overline{p^{-\nu} \frac{dG}{dx}} \equiv 1 \pmod{x}$. We conclude by Lemma 8.13 and equation (8.6). □

9. p -JETS OF $E[p^\nu]$ FOR SUPERSINGULAR ELLIPTIC CURVES

Theorem 9.1. *Let E/R be an elliptic curve with supersingular reduction and $E[p^\nu]$ the kernel of the multiplication by p . Then for any $n \geq 1$ we have:*

$$(9.1) \quad \mathcal{O}^n(\widetilde{E[p^\nu]}) = \frac{k[x, x', \dots, x^{(n)}]}{(x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n)})^{p^\nu})} \quad \text{if } n \geq 1,$$

$$(9.2) \quad \overline{\mathcal{O}^n(E[p^\nu])} = \frac{k[x, x', \dots, x^{(n)}]}{(x^{p^\nu}, (x')^{p^\nu}, \dots, (x^{(n-\nu)})^{p^\nu})} \quad \text{if } n \geq \nu,$$

$$(9.3) \quad \overline{\mathcal{O}^n(E[p^\nu])} = \frac{k[x, x', \dots, x^{(n)}]}{(x^{p^\mu})} \quad \text{if } 1 \leq n \leq \nu - 1,$$

where $\mu \geq \nu$.

Proof. Since E has supersingular reduction $E[p^\nu]$ is connected so it is equal to $\mathcal{F}[p^\nu]$ where \mathcal{F} is the formal group law of E and we conclude by Theorem 8.3. □

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